Solving Quadratic Equations by Completing the Square

Generally, one would solve quadratics by completing the square if the quadratic was prime (unable to factor). When solving by completing the square, first complete the square then use the square root property.

<u>Steps to solve by completing the square</u>:

1.) If the leading coefficient (coefficient of squared term) is not 1, divide all terms by the value of the leading coefficient.
2.) Collect the quadratic and linear terms on one side and the constant term on the other side.
3.) To find the value that completes the square, divide the linear (x) term's coefficient by 2, and square the result. Add this value to both sides.
4.) Factor the perfect square trinomial, and combine constants on the other side.
5.) Solve the equation with the square root property.
6.) Check by substituting the solution(s) back into the original equation.

Examples: Solve by completing the square.

a) $x^2 + 6x = -8$

Solution:

Step 1: Leading coefficient is 1, so step1 is complete.
Step 2: The quadratic and linear term are already collected on one side leaving the constant on the other, so step 2 is complete.

Step 3: The linear term is $6x$, which has a coefficient of 6. Dividing by 2, we obtain $\frac{6}{2} = 3$, now square it to get $(3)^2 = 9$

Adding 9 to both sides, you get: $x^2 + 6x \mathbf{+ 9} = -8 \mathbf{+ 9}$

$$\boldsymbol{x^2 + 6x + 9 = 1}$$

Step 4: Factor the left-hand side of the equation (it should be a perfect square trinomial!).

$$\boldsymbol{(x+3)(x+3) = 1}$$
$$\boldsymbol{(x+3)^2 = 1}$$

Step 5: Solve using square root property.

$$\boldsymbol{\sqrt{(x+3)^2} = \pm\sqrt{1}}$$
$$\boldsymbol{x + 3 = \pm 1}$$
$$\boldsymbol{x = -3 \pm 1}$$
$$\boldsymbol{x = -4,\ -2}$$

Step 6: Check!

$\boldsymbol{x = -4}$

$\boldsymbol{(-4)^2 + 6(-4) = -8}$

$\boldsymbol{x = -2}$

$\boldsymbol{(-2)^2 + 6(-2) = -8}$

$\mathbf{16 - 24 = -8}$
Checks!

$\mathbf{4 - 12 = -8}$
Checks!

Final Solution: $\boldsymbol{x = -4, -2}$

b) $x^2 - 7x - 8 = 0$

Solution:

Step 1: Leading coefficient is 1, so step1 is complete.
Step 2: Isolate the quadratic term and linear terms on one side by moving the constant to the other side of the equal sign.

$$x^2 - 7x = 8$$

Step 3: The linear term is $-7x$, which has a coefficient of -7. Dividing by 2, we obtain $\frac{-7}{2}$, now square it to get $\left(\frac{-7}{2}\right)^2 = \frac{49}{4}$

Adding $\frac{49}{4}$ to both sides, you get: $x^2 - 7x + \frac{49}{4} = 8 + \frac{49}{4}$

$$\boldsymbol{x^2 - 7x + \frac{49}{4} = \frac{81}{4}}$$

Step 4: Factor the left-hand side of the equation (it should be a perfect square trinomial!).

$$\boldsymbol{(x - \tfrac{7}{2})(x - \tfrac{7}{2}) = \tfrac{81}{4}}$$

(hint: you may have noticed the $-\frac{7}{2}$ is the same thing as what you get when you divide by 2 in step 3)

$$\boldsymbol{\left(x - \tfrac{7}{2}\right)^2 = \tfrac{81}{4}}$$

Step 5: Solve using square root property.

$$\boldsymbol{\sqrt{\left(x - \tfrac{7}{2}\right)^2} = \pm\sqrt{\tfrac{81}{4}}}$$
$$\boldsymbol{x - \tfrac{7}{2} = \pm\tfrac{9}{2}}$$
$$\boldsymbol{x = \tfrac{7}{2} \pm \tfrac{9}{2}}$$
$$\boldsymbol{x = \tfrac{7}{2} + \tfrac{9}{2} = 8 \quad \text{and} \quad x = \tfrac{7}{2} - \tfrac{9}{2} = -1}$$

Step 6: Check!

$\mathbf{x = 8}$
$\mathbf{(8)^2 - 7(8) - 8 = 0}$
$\mathbf{64 - 56 - 8 = 0}$
$\mathbf{0 = 0}$
Checks!

$\mathbf{x = -1}$
$\mathbf{(-1)^2 - 7(-1) - 8 = 0}$
$\mathbf{1 + 7 - 8 = 0}$
$\mathbf{0 = 0}$
Checks!

Final Solution: $\boldsymbol{x = -1, 8}$

c) $4x^2 - 4x - 1 = 0$

Solution:

Step 1: Leading coefficient is 4 and needs to be a 1, so divide everything by 4.

$\frac{4}{4}x^2 - \frac{4}{4}x - \frac{1}{4} = \frac{0}{4}$ simplifying it to: $x^2 - 1 - \frac{1}{4} = 0$

Step 2: Isolate the quadratic term and linear terms on one side by moving the constant to the other side of the equal sign.

$x^2 - x = \frac{1}{4}$

Step 3: The linear term is $-x$, which has a coefficient of -1. Dividing by 2, we obtain $\frac{-1}{2}$, now square it to get $\left(\frac{-1}{2}\right)^2 = \frac{1}{4}$

Adding $\frac{1}{4}$ to both sides, you get: $x^2 - x + \mathbf{\frac{1}{4}} = \frac{1}{4} + \mathbf{\frac{1}{4}}$

$$\boldsymbol{x^2 - x + \frac{1}{4} = \frac{1}{2}}$$

Step 4: Factor the left-hand side of the equation (it should be a perfect square trinomial!).

$\boldsymbol{(x - \frac{1}{2})(x - \frac{1}{2}) = \frac{1}{2}}$

(hint: you may have noticed the $-\frac{1}{2}$ is the same thing as what you get when you divide by 2 in step 3)

$\boldsymbol{\left(x - \frac{1}{2}\right)^2 = \frac{1}{2}}$

Step 5: Solve using square root property.

$\boldsymbol{\sqrt{\left(x - \frac{1}{2}\right)^2} = \pm\sqrt{\frac{1}{2}}}$

$\boldsymbol{x - \frac{1}{2} = \pm\frac{\sqrt{1}}{\sqrt{2}}}$ (note: square root of 1 is 1, and the denominator needs to be rationalized)

Rationalizing the denominator: $\boldsymbol{\pm\frac{\sqrt{1}}{\sqrt{2}} = \pm\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}}$

So, replace the right hand side with $\boldsymbol{\pm\frac{\sqrt{2}}{2}}$ to get

$\boldsymbol{x - \frac{1}{2} = \pm\frac{\sqrt{2}}{2}}$

$\boldsymbol{x = \frac{1}{2} \pm \frac{\sqrt{2}}{2}}$ **or can write** $\boldsymbol{x = \frac{1 \pm \sqrt{2}}{2}}$

Step 6: Check!

$\boldsymbol{x = \frac{1+\sqrt{2}}{2}}$

$\boldsymbol{4\left(\frac{1+\sqrt{2}}{2}\right)^2 - 4\left(\frac{1+\sqrt{2}}{2}\right) - 1 = 0}$

Using your calculator you should get $0 = 0$.

Checks!

Now check $x = \frac{1-\sqrt{2}}{2}$**,** it should work out as well.

Final Solution: $x = \frac{1\pm\sqrt{2}}{2}$